5. Shew that a definite meaning may be attached to the expression $\int_{-a}^{a} \frac{dx}{x}$ by supposing x to pass from $-a$ to $+a$ through a series of imaginary values, and that the result is $(2n+1)\pi\sqrt{-1}$, where n is an integer. Also give a geometrical illustration of the multiplicity of values.

6. Prove that the definite integral $\int_0^\infty \sin ax \,.\, x^{n-1}dx$ is convergent if n lies between the limits -1 and $+1$, and otherwise divergent. Also express the value of the integral in finite terms by means of the tabulated function $\Gamma(n)$.

7. A surface is referred to the polar co-ordinates r, θ, ϕ: required to express the direction-cosines of the normal at any point by means of the partial differential coefficients of r, the three lines to which the direction of the normal is referred being the radius vector, and two lines perpendicular to the radius vector, and lying respectively in and perpendicular to the plane in which θ is measured.

8. If u be a function of the rectangular co-ordinates x, y, z, and $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}$ be denoted by ∇u, and if T be a small space about the point (x, y, z), which in the end is supposed to vanish, U its volume, dS an element of its surface, dn an element of a normal drawn outwards, prove that

$$\nabla u = \text{limit of } \frac{\iint \frac{du}{dn}\, dS}{U}.$$

Hence obtain the expression for ∇u in polar co-ordinates, by taking for T the elementary volume $dr \,.\, rd\theta \,.\, r \sin\theta d\phi$.

9. A cylindrical wire is formed into a spring, which in its natural state has the form of a helix, the tangents to which are nearly perpendicular to its axis: given the radius of the cylinder in which the axis of the wire lies, and the modulus of torsion of the wire, find how much the spring will stretch when pulled by a small given force.

10. A body revolves in an ellipse of small eccentricity about a centre of force in one of the foci; examine the changes in the eccentricity and direction of the line of apsides produced (1) by a small central disturbing force acting when the body is in the neighbourhood of a given point of the orbit, (2) by a sudden small change in the absolute force, which takes place when the body is at that point.

Shew that the precise points in an orbit of any eccentricity at which the latter kind of disturbance would produce no change (1) in the distance between the foci, (2) in the ratio of that distance to the major axis, are (1) the extremities of the latus rectum passing through the second focus, (2) the extremities of the minor axis.

11. A sphere is held in contact with the interior surface of a rough hollow cylinder with its axis vertical, and is then projected horizontally in a direction parallel to the tangent plane at the point of contact; determine the motion.

12. If V be a function of the rectangular co-ordinates x, y, z which vanishes at an infinite distance from a given finite closed surface S, prove that

$$\iiint V \, . \, \nabla V \, . \, dx\, dy\, dz + \iint V \frac{dV}{dn} \, dS + \iiint \left\{ \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right\} dx\, dy\, dz = 0,$$

where the triple integrals extend to all infinite space outside S, and the double integral extends to the whole surface S, the meaning of ∇ and n being the same as in question 8.

13. If the normal component of the attraction of any mass be given throughout its surface, or else if the value of the potential throughout the surface be given, prove that the attraction on a particle external to the surface will be determinate, the law of attraction being that of the inverse square of the distance.

14. Shew that the equality of pressure in all directions in a fluid, whether at rest or in motion, is a necessary consequence of the hypothesis that the mutual pressure of two adjacent portions of a fluid is normal to the surface of separation.

Mention any phenomena which you conceive either to confirm the accuracy or to prove the inaccuracy of the above hypothesis, considering separately the cases of rest and motion.

15. Find, according to the usual suppositions, the rate at which air is admitted into a partially exhausted receiver, through a small orifice; examine the result obtained by supposing the exhaustion perfect, and explain the paradox.

16. Prove that the velocity of propagation of a long wave of small height in a uniform canal of any shape is equal to that acquired by a heavy body in falling through a space equal to the area of a transverse section of the fluid divided by twice the breadth at the surface.

17. Shew that the differential equation to a ray of light propagated through a medium of variable density, which is symmetrical with respect to the plane of xy, which is that of the ray, is

$$V - U\frac{dy}{dx} = \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2},$$

where U, V denote the partial differential coefficients $\dfrac{d \log \mu}{dx}$, $\dfrac{d \log \mu}{dy}$, μ being the refractive index.

Apply this equation to shew that the astronomical refraction (r) can be expressed in a series of the form

$$A \tan z - B \tan z \sec^2 z \ldots\ldots,$$

where z is the apparent zenith distance.

18. What phenomena shew that the velocity of propagation of light in vacuum is the same for all colours?

19. If it be assumed that the true wave surface in a biaxal crystal is symmetrical with respect to three planes at right angles to one another, that it is expressed by an equation of the fourth degree, and that each of its principal sections consists partly of a circle, the radii of the three circles being different, shew that it is the wave surface of Fresnel.

By what experiments has the third assumption been confirmed with respect to light of each particular degree of refrangibility?

20. A prism is cut from a doubly refracting substance, and a pencil of light is transmitted through it, the plane of incidence being perpendicular to the edge: the angle of incidence, the angle of the prism, and the deviation of one of the emergent pencils being observed, obtain formulæ for determining the direction and velocity of the corresponding wave within the crystal, and put the formulæ in a convenient shape for numerical calculation.

January, 1851.

1. Find the number of polygons of n sides, in an extended sense of the word *polygon*, that are formed by n indefinite right lines in a plane, supposing no two of the lines to be parallel, and no three to meet in a point.

2. Three circles are described, each touching one side of a triangle and two sides produced, and a new triangle is formed by joining their centres. The new triangle is then treated like the first, and this process is carried on indefinitely. Prove that the triangles so formed tend indefinitely to become equilateral.

3. A rigid body is acted on at each element of a closed curve by a force in the direction of a tangent, and proportional to the length of the element. Prove that the system of forces is equivalent to a couple, acting in a plane for which the area enclosed by the projection of the curve is a maximum, and having a moment proportional to this maximum area.

4. Prove, in the manner of Newton (*Principia*, Lib. I. Prop. I.), that if a body move in any manner under the action of forces directed to any number of points in a fixed straight line, the volume described by the triangle which has a given portion of the line for base and the body for vertex varies as the time.

5. If the probability of the occurrence of an error lying between x and $x + dx$ in the observation of a certain quantity be proportional to $\epsilon^{-hx^2}dx$, shew that in the long run the mean of the squares of the errors of observation will be to the square of the mean error as π to 2.

6. If Ox, Oy, Oz, and Ox', Oy', Oz', be two systems of rectangular axes, and l, m, n, be the cosines of the angles $x'Ox$, $x'Oy$, $x'Oz$, l', m', n', the same for y', and l'', m'', n'', the same for z', prove that

$$l'' = mn' - m'n, \quad m'' = nl' - n'l, \quad n'' = lm' - l'm,$$

the positive directions of the axes being so chosen that the direction of revolution $x'y'z'x'$ ... is the same as $xyzx$ ....

7. Prove the formulæ

$$\int_s^\infty \cos\frac{\pi}{2}s^2 ds = N\cos\frac{\pi}{2}s^2 - M\sin\frac{\pi}{2}s^2, \quad \int_s^\infty \sin\frac{\pi}{2}s^2 ds = M\cos\frac{\pi}{2}s^2 + N\sin\frac{\pi}{2}s^2,$$

where $M = \dfrac{1}{\pi s} - \dfrac{1\,.\,3}{\pi^3 s^5} + \dfrac{1\,.\,3\,.\,5\,.\,7}{\pi^5 s^9} - \ldots, \quad N = \dfrac{1}{\pi^2 s^3} - \dfrac{1\,.\,3\,.\,5}{\pi^4 s^7} + \ldots$

8. If u be a function of the oblique coordinates x, y, z, and

$$Q = A\frac{d^2u}{dx^2} + B\frac{d^2u}{dy^2} + C\frac{d^2u}{dz^2} + 2D\frac{d^2u}{dydz} + 2E\frac{d^2u}{dzdx} + 2F\frac{d^2u}{dxdy},$$

shew that there exists a system of rectangular coordinates, x', y', z', and an infinite number of systems of oblique coordinates, for which Q takes the form

$$A'\frac{d^2u}{dx'^2} + B'\frac{d^2u}{dy'^2} + C'\frac{d^2u}{dz'^2}.$$

9. Eliminate by differentiation the transcendental function from the equation

$$y = 1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2^2\,.\,3} - \frac{x^6}{1\,.\,2^2\,.\,3^2\,.\,4} + \ldots$$

and thence prove that the large roots of the equation $y = 0$ form ultimately an arithmetic series having a common difference $\frac{\pi}{2}$.

10. Integrate the simultaneous equations

$$\frac{dx}{x + cy - bz} = \frac{dy}{y + az - cx} = \frac{dz}{z + bx - ay}.$$

11. Find the most general values of u, v, w which satisfy at the same time the five following partial differential equations:

$$\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz},$$

$$\frac{dv}{dz} + \frac{dw}{dy} = \frac{dw}{dx} + \frac{du}{dz} = \frac{du}{dy} + \frac{dv}{dx} = 0.$$

12. Explain why the achromatism of an object-glass consisting of two lenses cannot be rendered perfect. Illustrate by a figure the distribution of the foci of the various colours along the axis of a pencil when a compensation is effected as far as possible.

Supposing that you had it in your power to measure the refractive index of each kind of glass for any one or more of the principal fixed lines of the spectrum, what measures would you take, and what numerical values would you substitute for $\delta\mu$ and $\delta\mu'$ in the ordinary formulæ, so as to produce the best effect?

13. In a Kater's pendulum, supposing the times of vibration about the two axes to be slightly different, investigate an expression for the time of vibration which must be employed in the calculation, in order that the deduced length of the seconds' pendulum may be correct, the distance of the centre of gravity from either axis being supposed approximately known.

14. A rigid body at rest is struck in the direction of a line not passing through the centre of gravity: find the conditions under which the initial motion will be simply one of rotation.

15. A rigid body is suspended symmetrically by two fine parallel wires. A vertical line drawn through the centre of gravity is equidistant from the suspending wires, and is a principal axis of the body. The body being turned round this axis through a small angle, and then left to itself, it is required to find the time of a small oscillation, taking into account the force arising from the torsion of the wires.

16. A finite portion of an infinite mass of heterogeneous elastic fluid, acted on by no external forces, is in motion in any manner: apply the ordinary equations of motion of a fluid to prove that the increase of vis viva during the time dt is equal to $2Udt$, where

$$U = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) dx\,dy\,dz.$$

Prove also directly that the expression under the integral signs, multiplied by dt, is the work done during the elementary time dt by the expansion of the fluid occupying the elementary volume $dx\,dy\,dz$.

17. In the calculation of the phenomena of diffraction produced when a screen containing one or more apertures is placed before a lens through which a luminous point is viewed in focus, prove that the error arising from the neglect of the spherical aberration of a direct pencil, or the astigmatism of an oblique pencil, is a small quantity of the fourth order, the obliquities of the several rays being regarded as small quantities of the first order.

18. Explain the mode in which the wave-lengths corresponding to the principal fixed lines of the spectrum have been accurately measured, investigating the requisite formula.

19. A pure plane-polarized spectrum is analyzed, and the analyzer is turned till the light is extinguished. A plate of selenite, of such a thickness as to give a difference of retardation amounting to several waves' lengths, is then interposed, behind the analyzer, with its principal planes inclined at angles of 45° to the plane of primitive polarization. Deduce from theory the appearance presented as the analyzer is turned round through 90°.

20. A uniform flexible string of infinite length, subject to a given tension, is continually acted on at a given point by a given variable force in a transverse direction: determine the motion. Also examine specially the case in which the force is expressed by $c \sin nt$.

February, 1852.

[N.B. *Only one question is to be answered out of each pair.*]

1 A. ABC is a triangle inscribed in a circle, and AB, AC are produced to meet in D, E, the tangent at the extremity of the diameter passing through A; prove that a circle may be described about the quadrilateral figure $BDEC$.

1 B. O is a fixed, and P a variable point, and in OP, produced if necessary, a point I is taken such that $OP \cdot OI = a^2$: if I be called the *image* of P, shew that the angle of intersection of any two curves in space which intersect will be equal to the angle of intersection of their images.

2 A. In any telescope in which the refraction at the object-glass is centrical, prove that the magnifying power is equal to the ratio of the clear aperture of the object-glass to the diameter of the bright image in front of the eye-piece.

Would this image be formed if the telescope were directed to a single luminous point?

2 B. Shew how to find whether the equilibrium of an irregular floating body is stable or unstable.

3 A. A plane mirror is moved by clock-work about an axis parallel to its own plane and to the axis of the earth, at such a rate as to make one turn in 48 hours; shew that it will reflect the Sun's light in a fixed direction, changes of declination &c. being neglected.

What choice of directions does a heliostat of this construction afford? If the direction of the reflected light is to be horizontal, find for a given place and a given day its inclination to the meridian.

3 B. Prove that the attraction of a uniform spherical shell on an external particle varies inversely as the square of the particle's distance from the centre of the shell, the law of attraction being that of the inverse square of the distance.

From Newton's demonstration of this theorem, deduce a geometrical solution of the following problem: To divide the shell into two parts, by a plane perpendicular to the line joining the particle with the centre, such that the attractions of the segments may be in a given ratio.

4 A. Three straight lines in a plane A, B, C meet in a point, and three others A', B', C' also meet in a point; prove that in general (AA', BB'), (AC', CB'), and (BC', CA') will meet in a point, the symbol (AA', BB') denoting the straight line joining the point of intersection of A and A' with that of B and B'.

How many such systems exist when the first six lines are given?

4 B. From a fixed point O a perpendicular ON is let fall on the tangent plane at any point P of a curved surface, and from ON, produced if necessary, a length OP' is cut off, such that $ON \cdot OP' = k^2$, k being a given constant; prove that the locus of P' is a surface from which the first surface may be got back by the same construction, the points P, P' being corresponding points on the two surfaces.

5 A. When the sum of the series

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots$$

is known, shew how the sum of the series

$$a_0 A_0 + a_1 A_1 x + a_2 A_2 x^2 + a_3 A_3 x^3 + \dots$$

may be found, where a_0, a_1, $a_2 \dots$ are given multipliers which recur according to a certain cycle.

Example. Find the sum of the infinite series

$$\frac{x}{1} - \frac{x^4}{1 . 2 . 3 . 4} + \frac{x^7}{1 . 2 . 3 . 4 . 5 . 6 . 7} - \dots$$

5 B. Given the equations of a system of curves in space containing two arbitrary parameters, shew how to find whether the curves admit of being cut orthogonally by a system of curved surfaces; and how in that case to find the equation of the system.

Examples. Let the equations of the system of curves be

$$(1) \quad x = Ae^{-nz}, \quad y = Be^{nz};$$
$$(2) \quad x^2 + y^2 = A, \quad y = x \tan n (z - B);$$

where A, B are the arbitrary parameters.

6 A. Explain, in a manner as elementary as is consistent with rigour, the manner in which the rotation of the Earth may be exhibited by means of a pendulum.

6 B. A rigid body is suspended by a string, regarded as infinitely long and light, which is attached to a point in one of the principal planes through the centre of gravity: explain the nature of the motion which takes place when the body is slightly disturbed in the most general manner. Find also the time of oscillation, and the direction of the axis of rotation, in that kind of oscillation in which the point of attachment moves in a direction perpendicular to the principal plane.

Form the quadratic which determines the time of oscillation when the point of attachment has any arbitrary position. Find also in this case the direction of the axis about which there must be no initial angular velocity in order that there may be no continual revolution of the body.

7 A. Water is flowing gently and regularly through a cylindrical pipe inclined at an angle α to the horizon: supposing the fluid to be retarded by an internal friction producing a tangential pressure proportional to the rate of sliding, and just counteracting the effect of gravity, determine the motion, supposing the film of fluid immediately in contact with the pipe to be at rest. Shew also that the mean velocity in the pipe is to the mean velocity in a broad stream flowing over a perfectly even bed having a uniform slope β as $3a^2 \sin \alpha$ to $8b^2 \sin \beta$, where a is the radius of the pipe, and b the depth of the stream, measured in a direction perpendicular to its bed.

Is the motion of running water in practical cases of the kind here supposed? If not, what do you conceive to be the cause of the difference, and what the nature of the actual motion?

7 B. An infinite uniform stretched string, not acted on by gravity, is loaded at a certain point (taken for origin) with a given mass. An indefinite series of small transverse disturbances, expressed by

$$c \sin k\,(at + x),$$

is continually propagated from an infinite distance, and is incident on the mass: determine the motion.

Determine also the simultaneous motions of the string and the mass when the latter is acted on by a given small disturbing force f, and the former is subject to no disturbances except those which travel from the mass outwards. Examine in particular the case in which $f = c \sin kat$ from $t = -\infty$ to $t = 0$, and $f = 0$ from $t = 0$ to $t = \infty$.

8 A. Give a general explanation of the fringes seen about the shadow of an opaque body bounded by a straight edge; of the continual and rapid decrease of illumination on receding from the geometrical shadow inwards; of the fluctuations of illumination outside, and of the increasing rapidity and decreasing amount of those fluctuations on receding from the geometrical shadow.

Shew that the fringes, regarded as existing in space, and considered only in a plane drawn through the luminous point perpendicular to the diffracting edge, form a system of hyperbolas starting from the edge; and find the law according to which the breadth of a given fringe depends upon the distances from the luminous point to the diffracting edge, and from the latter to the screen on which the fringe may be supposed to be received.

How did Fresnel measure the distances of the fringes from the geometrical shadow?

8 B. Shew that a tube filled with sirup of sugar, followed by a Fresnel's Rhomb, when interposed between a polarizing plate and double-image prism will present the same general phenomena as a plate of selenite without the rhomb, so far as regards the changes of colour produced by turning the prism round, but that the actual tints seen in the two cases will not precisely correspond.

How may the accuracy of the rhomb be tested by this arrangement?

February, 1853.

1. Through any point P in the diagonal AC of a parallelogram $ABCD$ are drawn any two straight lines meeting the sides AB, AD respectively, in E, F, and the opposite sides in G, H; prove that EF is parallel to GH.

2. Prove that the six planes bisecting the dihedral angles of a triangular pyramid meet in a point.

3. Examine the effect of the disturbing force of S on the nodes of P's orbit. (NEWTON, *Princip.* Lib. I. Prop. 66.)

4. When a body describes an ellipse round a centre of force in one of the foci, if a line be drawn from a fixed point always parallel to the direction of motion, and proportional to the velocity of the body, the extremity of the line will trace out a circle.

5. Given the specific heat of a gas when (1) the pressure p (2) the density ρ is constant, find the specific heat when p and ρ vary together in such a manner that $\frac{dp}{d\rho}$ has a given value; and interpret the result when $\frac{dp}{d\rho}=\frac{p}{\rho}$.

6. If an indefinitely small spherical portion of a homogeneous incompressible fluid in motion be suddenly solidified, shew that in addition to its motion of translation it will revolve round its centre with an angular velocity of which the components are

$$\tfrac{1}{2}\left(\frac{dw}{dy}-\frac{dv}{dz}\right),\quad \tfrac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dx}\right),\quad \tfrac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right).$$

7. Assuming the laws of reflexion and refraction, shew that in the case of a ray reflected or refracted any number of times $\Sigma\mu s$ is ultimately constant, when the points of incidence on the several surfaces are made to vary arbitrarily by indefinitely small quantities, s denoting the length of that portion of the path which lies within the medium whose refractive index is μ.

Apply this principle to find the general equations of a refracted ray, when a pencil of rays emanating from a given point is incident on a surface bounding a medium of different refractive power.

8. Give a full explanation of the formation of the primary and secondary rainbows on the principles of geometrical optics, and state any circumstances of the phenomenon which require for their explanation a more refined theory.

9. Two surfaces touch each other at the point P; if the principal curvatures of the first surface at P be denoted by $a \pm b$, those of the second by $a' \pm b'$, and if ϖ be the angle between the principal planes to which $a+b$, $a'+b'$ refer, δ the angle between the two branches at P of the curve of intersection of the surfaces, shew that

$$\cos^2\delta = \frac{a^2+a'^2-2aa'}{b^2+b'^2-2bb'\cos 2\varpi}.$$

10. A function is tabulated for a series of equidifferent values of the variable; investigate a formula for deducing the value of the differential coefficient of the function for a value of the variable intermediate between those found in the tables.

Ex. Given $u=1{\cdot}8733395$, $1{\cdot}8748744$, $1{\cdot}8764069$, $1{\cdot}8779372$, for $x=6{\cdot}51$, $6{\cdot}52$, $6{\cdot}53$, $6{\cdot}54$, find $\frac{du}{dx}$ for $x=6{\cdot}514$.

11. Integrate the differential equation

$$axdy^2+bydx^2=(xdy-ydx)^2,$$

and find its singular solutions.

12. Explain the method of integrating the partial differential equation

$$L\frac{du}{dx} + M\frac{du}{dy} + N\frac{du}{dz} + \ldots = V,$$

where $L, M, N \ldots V$ are functions of $x, y, z \ldots u$. Illustrate your explanation, in the case of two independent variables, by reference to geometry.

If $$x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} + \ldots = nu,$$

shew that u is a homogeneous function of $x, y, z \ldots$ of n dimensions.

13. Shew that

$$\int_0^\infty \sin ax \left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \ldots\right\} = 0 \text{ or } = \frac{1}{\sqrt{a^2 - 1}}$$

according as a, which is supposed to be positive, is less or greater than 1.

14. Compare the quantities of heat received from the Sun during one day at different places on the Earth's surface when the Sun has a given north declination, supposing the whole or a given fraction of the incident rays to be absorbed.

Shew that this quantity has two maxima, one at a latitude north of that at which the Sun is vertical at noon, the other at the north pole. Shew also that when the Sun's declination exceeds 17° 39′ more heat is received at the north pole than at the equator.

Given

$\cot^{-1}\pi = 17° 39'$; $\cos 2h = -\tan^2 25° 0'$, when $\tan 2h = 2h$ and $0 < 2h < 270°$.

15. Given the direction of an extraordinary ray within a crystal of Iceland spar, determine by a geometrical construction the direction of the bounding plane by which it will emerge without deviation. Find also, for a given bounding plane, the direction of the incident ray when the course of the extraordinary ray is a prolongation of that of the incident ray.

16. A heavy vertical circle is mounted so as to admit of sliding in a vertical plane down a smooth inclined plane, but is supported. A uniform smooth heavy string, whose length is less than half the circumference, is then laid on the circle so as to rest in equilibrium. The support being now suddenly removed, find at the commencement of the motion the tension of the string at any point and the pressure on the curve.

17. A heavy elastic string, uniform in its natural state, is attached at one extremity to a fixed point, and at the other sustains a heavy particle, and the system performs periodic oscillations in a vertical direction; shew that the time of oscillation is determined by a transcendental equation of the form $x \tan x = c$.

Discuss the roots of this equation, and shew how any required root may readily be calculated with the assistance of trigonometrical tables.

February, 1854.

1. Straight lines AP, BP pass through the fixed points A, B, and are always equally inclined to a fixed line; shew that the locus of P is a hyperbola, and find its asymptotes.

2. A number of equal vessels communicate successively with each other by small pipes, the last vessel opening into the air. The vessels being at first filled with air, a gas is gently forced at a uniform rate into the first; find the quantity of air remaining in the nth vessel at the end of a given time, supposing the gas and air in each vessel at a given instant to be uniformly mixed.

3. Separate the roots of the equation

$$2x^3 - 9x^2 + 12x - 4 \,.\, 4 = 0,$$

and find the middle root to four places of decimals by Horner's method, or by some other.

4. Investigate a formula in Finite Differences for transforming a series the terms of which (at least after a certain number) are alternately positive and negative, and decrease slowly, into one which is generally much more rapidly convergent.

Example. Find the sum of the series

$$1{\cdot}4142 - {\cdot}7071 + {\cdot}5303 - {\cdot}4419 + {\cdot}3867 - {\cdot}3480 + {\cdot}3190 - {\cdot}2962 + \ldots$$

5. Given the centre and two points of an ellipse, and the length of the major axis, find its direction by a geometrical construction.

6. Integrate the differential equation

$$(a^2 - x^2)\, dy^2 + 2xy\, dy\, dx + (a^2 - y^2)\, dx^2 = 0.$$

Has it a singular solution?

7. In a double system of curves of double curvature, a tangent is always drawn at the variable point P; shew that, as P moves away from an arbitrary fixed point Q, it must begin to move along a generating line of an elliptic cone having Q for vertex in order that consecutive tangents may ultimately intersect, but that the conditions of the problem may be impossible.

8. If X, Y, Z be functions of the rectangular co-ordinates x, y, z, dS an element of any limited surface, l, m, n the cosines of the inclinations of the normal at dS to the axes, ds an element of the bounding line, shew that

$$\iint \left\{ l\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + m\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + n\left(\frac{dY}{dx} - \frac{dX}{dy}\right)\right\} dS$$

$$= \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds,$$

the differential coefficients of X, Y, Z being partial, and the single integral being taken all round the perimeter of the surface*.

[* This fundamental result, traced by Maxwell (*Electricity*, I, § 24) to the present source, has of late years been known universally as Stokes' Theorem. The same

9. Explain the geometrical relation between the curves, referred to the rectangular co-ordinates x, y, z, whose differential equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

and the family of surfaces represented by the partial differential equation

$$P\frac{dz}{dx} + Q\frac{dz}{dy} = R.$$

10. Write a short dissertation on the theoretical measure of mass. By what experiments did Newton prove that masses may be measured by their weights? Independently of such experiments, how may it be inferred from the observed motions of the heavenly bodies that the mutual gravitation of two bodies depends only on their masses, and not on their nature? In what two different senses is the term *weight* used?

11. What are the conditions to be satisfied in order that two moving systems may be dynamically as well as geometrically similar?

If it be desired to investigate the resistance to a canal boat moving 6, 8, 10 miles an hour by experiments made with a small model of the boat and canal, if the boat be 36 feet long and its model only 2 ft. 3 in., what velocities must be given to the latter?

12. A rod is suspended at two given points by unequal light elastic strings, in such a manner that the rod is horizontal and the strings are vertical in the position of equilibrium; the rod being slightly disturbed in a vertical plane, in such a manner that no displacement or velocity is communicated to the centre of gravity in a horizontal direction, it is required to determine the motion.

13. Shew how to determine the time of rotation of the Sun about his own axis, and the position of his equator.

14. Rays coming from a luminous point situated in the axis of a large convex lens, and beyond the principal focus, are received after transmission through the lens on a screen held perpendicular to the axis, which is moved from a little beyond the extremity of the caustic surface to a little beyond the geometrical focus: compare, according to the principles of geometrical optics, the illumination at different points of the screen, and at different distances of the screen from the lens, the lengths of any lines in the figure being regarded as known.

kind of analysis had been developed previously in particular cases in Ampère's memoirs on the electrodynamics of linear electric currents. And in a letter from Lord Kelvin, of date July 2, 1850, relating to such transformations, which has been found among Stokes' correspondence, the theorem in the text is in fact explicitly stated as a postscript. The vector which occurs in the surface integral had been employed by MacCullagh, who recognized its invariance, about 1837, in optical dynamics. It reappeared in Stokes' hands in 1845 (*ante*, Vol. I, p. 81) as twice the differential rotation in the theory of fluid motion and formed the basis of the mathematical theory of viscosity of fluids, as also at a later time of Helmholtz's theory of vortex motion: its application to vibrations of solid elastic media was developed by Stokes in 1849 (*ante*, Vol. II, p. 253). Thus the theorem, though first stated by Lord Kelvin, relates to a quantity which, as regards physical applications, may be claimed to be Stokes' vector.]

Give a sketch of the method of finding the illumination in the neighbourhood of a caustic according to the theory of undulations. What is the general character of the result, and in what natural phenomenon is it exhibited?

15. A glass plate, the surface of which is wetted, is placed vertically in water; shew that the elevation of the fluid varies as the sine of half the inclination of the surface to the horizon, and compare its greatest value with the elevation in a capillary tube of given diameter. Find also the equation of the surface.

16. Explain the different modes of determining the Mass of the Moon.

17. Plane polarized light is transmitted, in a direction parallel to the axis of the crystal, across a thick plate of quartz cut perpendicular to the axis, and the emergent light, limited by a screen with a slit, is analyzed by a Nicol's prism combined with an ordinary prism; describe the appearance presented as the Nicol's prism is turned round, and from the phenomena deduce the nature of the action of quartz on polarized light propagated in the direction of the axis.

January 30, 1855.

1. A straight line drawn through the middle point of one side of a triangle divides the two other sides, the one internally and the other externally, in the same ratio.

2. If P, Q' be the points in one of two concentric and confocal ellipsoids which correspond to the points P', Q respectively in the other, prove that $PQ' = P'Q$.

3. Shew how to find whether $f(x, y, z) = 0$ is or is not a particular case of $u = \phi(v)$, where u and v are given functions of x, y, z.

4. A long cylinder, loaded on one side, rests with its axis not horizontal on a rough inclined plane; find the condition that equilibrium may be just possible.

5. Find the volume of the solid generated by the revolution of the closed part of the curve $x^3 - 3axy + y^3 = 0$ about the line $x + y = 0$.

6. If A, B be two terrestrial stations at no great distance, find approximately the difference between the north azimuth of B as seen from A, and the south azimuth of A as seen from B.

7. Two smooth perfectly elastic balls are suspended by threads from two points in the same horizontal plane, in such a manner that when the balls are at rest their centres lie in a horizontal plane, and the surfaces are just in contact. If the balls be withdrawn in different planes from their position of rest, and let go simultaneously, shew that the motion will be periodic, the effect of the rotatory inertia of the balls and of the finite arc of oscillation on the time of vibration being neglected.

8. Find the condition of achromatism of an eye-piece composed of a single block of glass worked at the ends into spherical surfaces, and used with an object-glass of great focal length.

9. Given the force of gravity at the top and bottom of a mine of known depth, and the density of the intervening mass, find the mean density of the Earth.

10. If from a variable point in a conic section perpendiculars be let fall on the sides of any fixed, inscribed polygon with an even number of sides, the product of the perpendiculars let fall on one set of alternate sides will be to the product of the perpendiculars let fall on the other set in a constant ratio.

11. A wheel, of which an axle projecting on each side forms a part, is supported in a vertical plane by having the axle on each side resting on a pair of friction wheels, each of which is just like the first wheel, and is similarly supported, and so on indefinitely; compare the inertia of the whole system, in relation to a rotation of the first wheel, with that of the first wheel alone.

12. Find in any manner the approximate effect of the Sun's disturbing force on the mean motion of the Moon in a month, and thence deduce the Annual Equation.

13. If a body move round the Sun in a circular orbit, the radius of which is slowly diminishing in consequence of a resisting medium, find the amount of work spent on the medium as the body moves from one given distance to another; and compare the amount spent in passing from an infinite distance to close to the surface with the work spent when the body impinges on the Sun, and is thereby reduced to rest.

14. Shew that

$$\int_{-\infty}^{\infty} e^{-\left(x^2\cos 2\theta+\frac{a^2}{2x^2}\sin 2\theta\right)} \begin{matrix}\cos\\ \sin\end{matrix}\left\{x^2\sin 2\theta+\frac{a^2}{2x^2}\cos 2\theta\right\}dx=\pi^{\frac{1}{2}}e^{-a}\begin{matrix}\cos\\ \sin\end{matrix}(\theta+a),$$

θ being comprised between the limits $\pm\frac{\pi}{4}$.

15. Investigate the series of notes which may be produced by the vibration of the air within a tube closed at one end and open at the other. Why is the supposition usually made as to the condition at the open end inexact, and in what manner do the results of theory, on this supposition, differ from those of experiment?

16. Find numerically what would be the ellipticity of the Earth if it consisted of a sphere in which the density was a function of the distance from the centre, covered by a comparatively shallow superficial crust having a density equal to $\frac{5}{11}$ths of the mean density.

17. Under what experimental circumstances is the reflected system of Newton's rings white-centred, either with common or polarized light, and what point of theory in each case is confirmed by the phenomenon?

18. A solid of revolution is made to revolve with great rapidity round its axis, which is so mounted that the solid is free to turn in all directions round its centre of gravity. If a force be applied perpendicularly to the axis, a considerable resistance is experienced, and the instantaneous axis moves perpendicularly to the direction of the force; but if the axis be mounted so as to be moveable in one plane only, and a force be applied in that plane, the solid goes round just as if it had no motion of rotation originally. Explain this.

19. In a system of curves in space whose equations contain two arbitrary parameters, shew that the pencil of tangents drawn at the points where the curves are cut by a small plane perpendicular to one of them consists ultimately of a pencil of parallel lines altered (1) by being made to converge to two focal lines in rectangular planes, (2) by being twisted; and that the analytical condition of the possibility of cutting the lines orthogonally by a system of surfaces expresses that there is no twisting. Shew also geometrically how the twisting would render it impossible to cut the lines in the manner described.

20. Write a short dissertation on the evidence, or want of evidence, of the truth of the laws of double refraction in biaxal crystals which result from the theory of Fresnel. Do you conceive these laws to be rigorously or only approximately true; to be applicable to each colour in particular, or only to white light as a whole?

January 30, 1856.

1. If the distances of the points A, B, C, D in a straight line from a point O in the same line be in harmonic progression, prove that the rectangle under the extreme segments AB, CD of AD is equal to the rectangle under the middle segment and the mean of the three.

2. Required a point which is at the same time the middle point of each of two chords of two given circles respectively, these chords both passing through the same given point. Is the solution of the problem always geometrically possible?

3. Are there any objections to the representation of imaginary branches of a curve by branches lying in a perpendicular plane?

Shew that the asymptotes of a given circle are independent of the arbitrary choice of rectangular co-ordinate axes.

4. Trace the curve

$$\left(\frac{x-a}{x}\right)^2 + \left(\frac{y-b}{y}\right)^2 = 1.$$

5. Shew that the family of curves

$$\left(\frac{x^2}{c^2-a^2} + \frac{y^2}{c^2-b^2}\right)^2 = \frac{4}{c^2}\left(\frac{a^2x^2}{c^2-a^2} + \frac{b^2y^2}{c^2-b^2}\right)$$

has the same envelope whether a or b be the variable parameter.

How do consecutive curves of the system lie when the points of contact with the envelope are imaginary?

6. The equation of a surface is given implicitly by explicit expressions for x, y, z in terms of two parameters; required the expressions for the direction-cosines of the normal.

7. Find the lines of magnetic force in the case of a pair of poles of equal strength, regarded as points, (1) when the poles are of the same name, (2) when they are of opposite names. Shew how the lines may be graphically constructed.

8. If a triangle be circumscribed about a conic, and the points of contact be joined with the opposite vertices, and tangents drawn at the points of intersection with the conic so as to form a second triangle, prove that the relation between the two triangles will be reciprocal.

9. Shew how to find the lines on a surface at which one of the principal curvatures vanishes. How does the form of the surface in general alter in passing across such a line? Is the line in question a line of curvature?

10. If $f(x) = x^n + p_1 x^{n-1} \dots + p_{n-1}x + p_n$, where $p_1 p_2 \dots p_n$ are imaginary, and if $f(x + \sqrt{-1}y) = P + \sqrt{-1}Q$, where P and Q are real, examine the forms of the intersections of the surfaces $z = P$, $z = Q$ with the plane $z = 0$, and thence shew that the equation $f(x) = 0$ has n roots of the form $x + \sqrt{-1}b$.

11. What is meant by the *complete primitive*, the *general primitive*, and the *singular primitive equation* of a partial differential equation of the first order between three variables? Illustrate the relation between these three forms by reference to geometry.

Explain the mode of integrating the equation $f(x, y, z, p, q) = 0$.

12. What is meant by the irrationality of dispersion, and how are its effects manifested in an object-glass? Shew that they may be got rid of (to the lowest order of approximation) by combining a crown-glass lens with a flint-glass lens of proper strength, placed a proper distance down the tube of the telescope. What would be the objection to such a construction?

13. Two rods equally inclined to the vertical meet in an angle, which opens upwards; required the smallest angle of a double cone which if laid symmetrically upon the rods will roll away from the angle.

14. Find the latitude at sea from two altitudes of the Sun and the time between, correcting for the ship's change of place.

15. Explain why a solid iron plate should be stronger than a compound plate made up of thinner plates rivetted together, and having the same aggregate thickness.

16. Two equal streams of light interfere, but the difference of path is too great to allow of the exhibition of colours. The mixed streams being limited by a screen with a slit parallel to the direction which the fringes would have if they appeared, and being then analyzed by a prism, explain the appearance presented; and shew

how to deduce from observation the absolute retardation, supposing the wave-lengths for two given points of the spectrum to be known, and the difference of path to have occurred in air.

17. What is the relation between the invariable plane of a dynamical system and the set of impulsive forces by which the system may have been originally set in motion? Would there exist an invariable plane in the case of a system composed of the fragments of a body which exploded having been previously at rest?

A homogeneous ellipsoid at rest, fixed at its centre, being struck in the direction of a given line, find the angle between the initial instantaneous axis and the normal to the invariable plane.

18. Obtain the equation of steady motion in hydrodynamics, shewing clearly in what sense and under what restrictions it is true.

19. Shew that the effect of the term $\frac{15}{4} me \sin\{(2-2m-c)pt-2\beta+\alpha\}$ in the expression for the Moon's longitude is equivalent to that of a periodic fluctuation in the excentricity and longitude of the perigee. Account for these fluctuations by general reasoning.

20. A ray is incident in a given direction on a doubly refracting medium; explain clearly the steps of the physical reasoning which leads to a geometrical construction for determining the directions of the refracted rays when the velocity of plane waves within the medium is known as a function of the direction.

February 3, 1857.

1. If $a > b > c$, then $\frac{a}{c} + \frac{b}{a} + \frac{c}{b} > \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$.

2. Shew that every imaginary plane contains one and but one real straight line.

3. Prove that the mean of n positive quantities which are not all equal is greater than the nth root of their product.

4. When the sum of a series according to ascending powers of x is known, shew how to deduce the sum of the series obtained by taking terms of the former at regular intervals.

Example. From the series

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

deduce the value of $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots$

5. Obtain a series for the ready calculation of $\int_x^\infty e^{-x^2} dx$ for large values of x.

6. Given that

$$y\frac{dz}{dx} = x\frac{dz}{dy}, \text{ and } cz = (x-y)^2 \text{ when } x+y=a,$$

required the relation between x, y, z.

7. Trace the curve

$$(x+y)^2(x^2+y^2) - a^3x - b^3y = 0,$$

and shew how it passes into what it becomes when $b = a$.

8. If the equation of a curve be given implicitly by two equations of the form $x = \phi(t)$, $y = \psi(t)$, where $\phi(t)$, $\psi(t)$ are free from radicals, shew how to find (1) the double points; (2) the cusps, regarded as singular double points; (3) the cusps, by a method not introducing the double points.

Apply your method to the example

$$a^2x = (t+a)^2(t-b), \qquad b^2y = (t+b)^2(t-a).$$

9. Explain the generation of a developable surface (1) by lines, (2) by planes of which it is the envelope; and point out the mutual relations of the generating lines, the enveloping planes, and the cuspidal edge. Explain also the form of the surface about such an edge.

10. Explain the effect of the sun's disturbing force in rendering the moon's orbit oval, supposing the undisturbed orbit circular; and find the ratio of the axes of the oval orbit.

11. A uniform flexible string is suspended by one end, and performs small movements in a vertical plane; form the differential equation of the motion, and thence obtain, in a series, the transcendental equation whose roots determine the periodic times of the various possible symmetric oscillations.

12. Obtain the equations of motion of a rigid body moveable about a fixed point, in terms of the angular velocities about the principal axes through that point.

Prove that the rotation of a free body revolving round a principal axis through its centre of gravity is stable or unstable according as that axis is one of greatest or least, or else one of mean, moment of inertia.

13. Shew that a small lunar atmosphere would affect the duration of an occultation, but not sensibly affect the apparent diameter of the moon.

14. How do you account for the remarkable effect of wind on the intensity of sound?

15. Find the condition of achromatism of an eye-piece composed of a solid block of glass with a thin lens of different glass cemented to it on the end next the eye, the surfaces being worked spherical.

16. Find the attraction of a prolate spheroid on an internal particle.

A mass of homogeneous fluid is subject to the mutual gravitation of its particles, and to a repulsive force tending from a plane through its centre of gravity and varying as the perpendicular distance from that plane; shew that the conditions of equilibrium will be satisfied if the surface be a prolate spheroid of a certain ellipticity, provided the repulsive force be not too great.

17. Account for the spectra formed by a fine grating; and supposing the grating placed obliquely to the incident light, find an expression giving the length of a wave of light in terms of quantities which may be observed.

18. If a continuous medium be continuously displaced, shew that the most general displacement of an element of the medium consists of a displacement of translation, the same as that of a point P taken in the element, a rotation round some axis through P, and three elongations along three rectangular axes passing through P.

19. Expressing the equations of motion of a fluid in a form in which the particle is supposed to remain the same in differentiations with respect to the time, and supposing the density either constant or a function of the pressure, and the forces such that $Xdx + Ydy + Zdz$ is a perfect differential, obtain first integrals of the three equations resulting from the elimination of the pressure. Are these equations altogether independent of each other? What important theorem may be proved by means of these integrals?

20. Describe fully some one experiment by which it may be shewn that two streams of light from the same source, polarized in rectangular planes, and afterwards brought to the same plane of polarization, do or do not interfere according as the light from the primitive source is or is not polarized.

N.B. It is not to be assumed that the colours of crystalline plates in polarized light are due to interference.

February 2, 1858.

1. If three circles pass each through one corner of a triangle and the points of bisection of the adjacent sides, they will meet in a point.

2. If XYZ be a spherical triangle having each side a quadrant, XPN, YPM great circles cutting the sides in N, M, and if the sides ZM, &c. of the quadrilateral $ZMPN$ be denoted by x, η, ξ, y, find the relations between x, y and ξ, η. If the sides of the quadrilateral be small, up to what order of small quantities will the opposite sides be equal?

3. Shew generally how to form, by elimination, the equation whose roots are the sums of every two roots of a given equation, and apply the general method to obtain the actual result in the case of the equation

$$x^3 + ax^2 + bx + c = 0.$$

4. In the single moveable pulley with weights not in equilibrium, find by direct application of D'Alembert's Principle the accelerating force and the tension of the string, neglecting the rotatory inertia of the fixed and of the moveable pulley.

5. A mirror of given aperture and focal length, and of small curvature, has the form of a prolate spheroid; shew that the aberration for parallel rays varies inversely as the major axis.

6. A sailor provided with a chronometer, sextant, and artificial horizon lands on a small island for a short time on a moonless night, the sky being cloudy towards the north; what observations would he take to determine the geographical position of the island? Reduce the deduction of the required from the observed quantities to the solution of spherical triangles, without writing down the formulæ.

7. Find the condition which must be satisfied in order that

$$Udx + Vdy + Wdz$$

may be the exact differential of a function of two independent variables, the equation connecting x, y, z being

$$pdx + qdy + rdz = 0,$$

which is supposed to be integrable as an equation between three independent variables.

8. Find the cusps of the curve defined by the equations

$$b^2 (a^2 - b^2) x^2 = (2b^2 - \theta)^2 (a^2 - \theta); \quad a^2 (a^2 - b^2) y^2 = (2a^2 - \theta)^2 (\theta - b^2),$$

where θ is the variable parameter.

9. If a curve be treated as defined by explicit expressions for the co-ordinates x, y involving a variable parameter, double points do not appear as singularities, but if by an equation $f(x, y) = 0$ they do. Account for this.

10. An infinite cylinder revolves uniformly round its axis in an infinite mass of fluid, which is thus made to revolve in cylindrical shells in a permanent manner; find the angular velocity of any shell, assuming that the tangential pressure arising from friction varies as the rate of sliding, and that the shell in contact with the cylinder revolves with it.

11. A chain fixed at two points to a vertical axis revolves uniformly about it; find the differential equation of the curve which it forms by the condition that the function which expresses the total work of the forces shall be a maximum, and shew how the arbitrary constants are to be determined.

12. If the particles of a continuous medium be slightly and continuously displaced, shew that the relative displacement about a given point is symmetrical with respect to a system of rectangular axes.